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Dynamical Problems of Ice Cover Fracture

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ABSTRACT

The problem of a crack in bending ice cover has been considered. The crack is assumed to be straight and semi-infinite, propagating under the action of a moving load. The ice cover is supposed to be an elastic plate contacting a heavy perfect liquid. Steady-state and resonance cases have been examined.

NOMENCLATURE

c	: sound velocity
C, S	: Fresnel integrals
D	: rigidity of the plate
E	: Young's modulus of elasticity
g	: gravity acceleration
H	: depth of liquid
h	: ice cover height
J_0	: Bessel function
$k = 2\pi/\lambda$	: wave number
L	: unit of length
K, q	: parameters of Fourier transforms with respect to η, y
K_*	: critical value of K
M	: bending moment
P	: pressure of liquid onto plate
Q	: external transverse load
s	: parameter of Laplace transform
R, θ	: polar coordinates on η, y plane
r, φ	: polar coordinates on K, q plane
t	: time
T	: energy spent for fracture
$u = v/\sqrt{gL}$	: nondimensional velocity
V	: mass velocity
V_x	: projection of V on x axis
v	: velocity of displacement of steady-state picture along x axis
v_*	: critical velocity
u_*	: nondimensional critical velocity
w	: transverse displacement ($w > 0$ for downward displacement)
w_0	: nondimensional displacement for $r = 0$
w_1	: nondimensional value of $\partial w/\partial x$ for $r = 0$
x, y	: Cartesian coordinates
z	: vertical coordinate directed downwards
δ	: Dirac's function
$\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$	: Laplace operator
ρ	: ice density
ρ_0	: liquid density
ν	: Poisson's ratio
$\Phi(t, x, y, z)$	: potential
λ	: wave length

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INTRODUCTION

The problem of dynamical crack propagation in ice cover (which is assumed to be a homogeneous elastic plate) differs from the classical problems of fracture mechanics in certain features. First, it is nonlocality, i.e., the action of forces bending the ice cover is transferred to its distant areas not only by the bending waves moving along the plate, but also by the waves in water. This fact considerably influences the mathematical formulation of the problem: Differential equations of dynamical bending of a plate contacting with water are transformed to a convolution-type equation (which makes its solution more difficult). That is why in this paper we shall first consider the fundamental problem of a normal moving force. Incidentally, solving this problem helps determine the feasibility of simplifications in the descriptions of plate-water interaction.

Secondly, the problem under consideration is characterized by critical velocity (i.e., velocity of movement of normal force and crack propagation), which is very small as compared to the corresponding critical velocity of a crack in an elastic medium (usually it is Rayleigh's waves velocity). The moving load induces waves carrying energy away "to infinity" if the velocity is greater than the critical one. In this case (in contrast to under-critical range) only a part of the energy is spent on fracture (flowing down a crack tip); another part is radiated by the waves mentioned above. The power of this radiation is to be included in correlation among the load intensity, velocity and location of a crack with respect to the load. This situation resembles the one of crack propagation in structured mediums (Slepian, 1985).

It should be added that the crack problem in a bent plate is really three-dimensional as there is no symmetry of deformation that orients a crack edge normally to the surface. So two-dimensional equations of plate bending become invalid in the vicinity of a crack edge. This obstacle can be avoided if the load is located far off the edge (in comparison with the plate thickness), where the influence of its deflection from the normal may be neglected. As for the conditions of fracture, i.e., crack propagation, in a steady-state problem the form of the crack edge is not essential, when Griffith's energy criterion is used.

The problem under consideration is a greatly idealized picture of real ice fracture (by ice-breakers, air-cushioned vessels, bridge bearings, etc.). However, the analysis may be useful for development of a theory of concrete processes in certain conditions, and for definition of the role (force or power) of main-line crack growth in the general picture of ice cover fracture.

DYNAMICAL EQUATIONS FOR ELASTIC PLATE LYING ON SURFACE OF IDEAL LIQUID

Though those equations (for the case of constant depth of liquid) are well-known (see Slepian, 1972; Hejsin, 1967), we give here a short deduction of main correlations and discuss the details that are important for the determination of solution of a steady-state problem.

Dynamical equation for the plate:

$$D\Delta^2 w + \rho h \ddot{w} = Q - P,$$
$$D = Eh^3 \Big/ \left(12\left(1-\nu^2\right)\right) \tag{1}$$

To this the dynamical equation for heavy liquid should be added. At first it is convenient to take into account its compressibility. Linearized equations for vorticity-free motion are (see Slepian, 1972; Lighthill, 1978):

$$\Delta\Phi - \frac{1}{c^2}\ddot{\Phi} = 0$$
$$V = grad\Phi, \qquad P = -\rho_0\dot{\Phi} + \rho_0 gz \tag{2}$$

These equations are correct if $|P| \ll \rho_0 c^2$, $|V| \ll c$. However, if the energy of compression of liquid is much less than its kinetic energy, i.e., if the process is stream-like, rather than that of wave propagation, the linearization of Cauchy-Lagrange equations (expressions for pressure in Eq. 2) may be invalid, though the linearization of wave equation is correct (Slepian, 1985). In this case:

$$P = -\rho_0\left(\dot{\Phi} + \frac{V^2}{2}\right) + \rho_0 gz \tag{3}$$

It is possible to estimate the convection term $\rho_0 V^2/2$ above. For this purpose consider the solution of a steady-state problem in which:

$$\Phi = \Phi(x - vt, y, z),$$
$$\dot{\Phi} = -v\frac{\partial\Phi}{\partial x} = -vV_x \tag{4}$$

It is obvious that linearization in Eq. 3 is valid, if:

$$|V| \ll v \tag{5}$$

For estimation of $|V|$ we assume w and Φ to be sinusoidal waves ($v^2 < c^2$):

$$w = w_0 \sin\left[k(vt - x)\right],$$
$$\Phi = \Phi_0 e^{-kz\sqrt{1-v^2/c^2}} \cos\left[k(vt - x)\right] \tag{6}$$

At the surface of the plate-water contact:

$$\frac{\partial\Phi}{\partial z} = \dot{w} \tag{7}$$

From Eqs. 6 and 7, we obtain:

$$\left|\frac{\partial\Phi}{\partial x}\right|_{\max} = \frac{kw_0 v}{\sqrt{1 - v^2/c^2}}$$
$$\left|\frac{\partial\Phi}{\partial z}\right|_{\max} = kw_0 w \tag{8}$$

Thus, if $v^2 \ll c^2$, which is characteristic for our problem, a stronger inequality

$$kw_0 \ll 1 \tag{9}$$

should be correct. It is obvious that this inequality results from the conditions of ice strength. It will be shown below that Eq. 9 is valid for such bending wave lengths, which are essential for description of the problem's solution.

Further we transform Eqs. 1 and 2 and condition (7) using Fourier transformations with respect to the variables x, y, and Laplace transform with respect to t with the parameters K, q, s, accordingly. We shall designate those transforms by the same letters, but with the arguments K, q, s.

Assume H as the depth of the liquid (then $z = H$ means the level of the bottom). After Fourier transformation, solving Eq. 2 as an ordinary differential one, from (7) and condition on the ground:

$$v_z = \frac{\partial\Phi}{\partial z} = 0, \qquad (z = H) \tag{10}$$

we can find that on the surface ($z = 0$)

$$P(s,K,q) = -\rho_0 s\Phi(s,K,q,0) =$$
$$= \rho_0 s^2 w(s,K,q)\frac{cth(H\ae)}{\ae},$$
$$\ae = \sqrt{K^2 + q^2 + s^2/c^2} \tag{11}$$

When H is finite the right part of Eq. 11 has no branch points. However, in the most interesting case of greater depths, when it is possible to assume coth $(H\ae) = 1$, there appears a problem of choosing a radical branch. Then, as it follows from the natural condition

$$\Phi \to 0, \qquad (z \to \infty) \tag{12}$$

it must be $\ae > 0$ if K, q, s are real and are not zero simultaneously.

Further we shall discuss transformations in a uniformly moving system of coordinates. For this purpose it is sufficient to let $s = \rho + iKv$, where ρ, K are parameters of Laplace transformation with respect to t and Fourier transformation with respect to $\eta = x - vt$ of functions of t, η (Slepian, 1972). Then radical $\ae$, when ρ, q are real, has two branch points, and if $v^2 < c^2$ then one point is in the upper half-plane, and the other is in the lower one of the complex plane. The points draw together while $q^2 + \rho^2/c^2 \to 0$. The branch of radical is being fixed by its above-mentioned value and by the cuts settled from the branch points up to infinity without crossing the real axis. In particular, when $q = \rho = 0$:

$$\ae = \sqrt{0 + iK}\sqrt{0 - iK} \tag{13}$$

For transition to the steady-state problem, using the Tauberian type theorem for Laplace transformation, it is sufficient in Eq. 11

to tend ρ to zero, i.e., to make $s = iKv + 0$ (another substantiation of the same rule is given by Lighthill, 1978). For our problem with comparatively low velocities ($0 \le v < c = 1500$ m/s), it is possible to let $c = \infty$, i.e., to neglect compressibility of the liquid. For $H = \infty$ (in coordinates η, y, where dependence of t is absent) it follows that:

$$P(K,q) = -\rho_0 \frac{(0+iKv)^2}{\ae} w(K,q) \qquad (14)$$

As it will become obvious below, keeping to the right limit in Eq. 14 is essential for supercritical speed analysis. Returning to Eq. 1, it is possible to get to the steady-state problem for infinite ice cover:

$$\left[D\left(K^2+q^2\right)^2 + \rho_0 g + (0+iKv)^2 \rho h + \right.$$
$$\left. + \frac{\rho_0 (0+iKv)^2}{\ae}\right] \cdot w(K,q) = Q(K,q),$$
$$\ae = \sqrt{K^2+q^2} \qquad (15)$$

Note that the densities of ice ρ and water ρ_0 are similar and product $\ae h \ll 1$. Otherwise Eq. 1 would be wrong; further we shall estimate this result. That is why the last but one member in the left part of Eq. 15 may be neglected in comparison with the last one. So we shall later on base on the equation:

$$L(K,q,v)\, w(K,q) = Q(K,q),$$
$$L(K,q,v) = D\left(K^2+q^2\right)^2 + \rho_0 g + \frac{\rho_0 (0+iKv)^2}{\sqrt{K^2+q^2}} \qquad (16)$$

SOLUTION OF PROBLEM FOR AXIALLY SYMMETRIC LOAD

Consider polar coordinates on the plane K, q, assuming:

$$K = r\cos\varphi \qquad q = r\sin\varphi;$$
$$x = R\cos\theta \qquad y = R\sin\theta,$$

if we take $Q(x, y) = Q_0(R)$, $R = \sqrt{x^2+y^2}$ from Eq. 16 it can be obtained

$$w = \frac{1}{2\pi}\int_0^\infty \int_0^{2\pi} \frac{Q(r) e^{-irR\cos(\varphi-\theta)}}{G(r,\varphi)} r^2 d\varphi\, dr,$$
$$\frac{\partial w}{\partial x} = -\frac{i}{2\pi}\int_0^\infty \int_0^{2\pi} \frac{Q(r) e^{-irR\cos(\varphi-\theta)}}{G(r,\varphi)} r^2 d\varphi\, dr,$$
$$G(r,\varphi) = Dr^4 + \rho_0 g + \rho_0 r (0 + iv\cos\varphi)^2,$$
$$Q(r) = \int\int_{-\infty}^{\infty} Q(x,y) e^{iKx+iKy} dxdy =$$
$$= 2\pi \int_0^\infty Q(R) J_0(r,R) R dR \qquad (17)$$

In particular, for a fundamental problem:

$$Q_0(R) = \delta(R) = \delta(x)\delta(y)$$

Note that the function $G(r, \varphi)$, when $v^2 < v_*^2$, does not become zero. In this subcritical range integrands in Eq. 17 are integrated ordinarily, so in the expression for G notation of the right-hand limit can be omitted. Thus, it is obvious that:

$$\frac{\partial w}{\partial x} = 0, \qquad (R=0)$$

It corresponds to the fact that, while the speed is subcritical, the load does not induce the waves that could take away energy, and hence it does not produce any work. Indeed, the power developed by the load is:

$$N = \frac{1}{2\pi}\int_0^\infty \int_0^{2\pi} Q_0(R) \frac{\partial w}{\partial t} R d\theta dR =$$
$$= -\frac{v}{2\pi}\int_0^\infty \int_0^{2\pi} Q_0(R) \frac{\partial w}{\partial x} R d\theta dR =$$
$$= -iv \int_0^\infty \int_0^{2\pi} \frac{Q^2(r)\cos\varphi}{G(r,\varphi)} r^2 d\varphi dr = 0 \qquad (18)$$

as the integral of φ is zero.

When the speed is overcritical, function G becomes zero at two closed contours at the plane r, φ (located symmetrically with respect to y axis). Contribution of these singularities to integral (18) is not zero (it can be calculated by means of the limit for $G(r, \varphi)$ mentioned above). When $v^2 > v_*^2$ the power, induced by the load, is being spent on stimulation of the bend-gravitation waves, carrying the energy up to infinity.

The double root $r = r_*$ of the equation $G(r, 0) = 0$ for $\varphi = 0$, $\varphi = \pi$ corresponds to critical value $v = v_*$.

Equations

$$G = Dr_*^4 + \rho_0 g - \rho_0 r_* v^2 = 0,$$
$$\frac{\partial G}{\partial r_*} = 4Dr_*^3 - \rho_0 v^2 = 0 \qquad (19)$$

determine critical values

$$v_* = 2 \cdot 3^{-3/8}\left(Dg^3/\rho_0\right)^{1/8},$$
$$r_* = \left(\rho_0 g/3D\right)^{1/4} \qquad (20)$$

From Eq. 17 it can be deduced that while in the point of loading

$$w = \frac{1}{2\pi\sqrt{\rho_0 g D}}\int_0^\infty Q(\xi L)\beta_1(\xi)d\xi,$$
$$\frac{\partial w}{\partial x} = 0 \qquad (21)$$

When $v^2 > v_*^2$

$$w = \frac{1}{2\pi\sqrt{\rho_0 g D}}\int_\Omega Q(\xi L)\beta_1(\xi)d\xi,$$
$$\frac{\partial w}{\partial x} = -\frac{1}{2\pi\sqrt{\rho_0 g D}}\int_w Q(\xi L)\beta_2(\xi)d\xi \qquad (22)$$

where

$$\beta_1(\xi) = \frac{\xi}{\sqrt{\left(\xi^4+1\right)\left(\xi^4+1-u^2\xi\right)}},$$
$$\beta_2(\xi) = \frac{\xi^2}{\sqrt{u^2\xi\left(u^2\xi-\left(\xi^4+1\right)\right)}},$$
$$L = \left(D/(\rho_0 g)\right)^{1/4}, \quad u = v/\sqrt{Lg}, \quad r = \xi L,$$
$$\xi \in \Omega, \quad \left(\xi^4+1 \ge u^2\xi\right)$$
$$\xi \in w, \quad \left(\xi^4+1 < u^2\xi\right)$$

RESONANCE CASE

No steady-state solution can correspond to critical value $v = v_*$. This is the case of resonance, i.e., unlimited growth of displacement during a period of time. (Other resonance wave phenomena and methods of their investigation are described in the 1972 book by Slepian).

To analyze the resonance case it is necessary to return to the transient problem, and to replace in Eq. 16 $0 + iKv$ by $\rho + iKv$ and $Q(K, q)$ by $Q(K, q)/\rho$.

To determine asymptotics of the solution for $t \to \infty$ it is enough to retain only the first power in the expression for L (16). Then by inversion of Laplace transform (in the moving system of coordinates), instead of Eq. 17 we obtain ($R = 0$):

$$w \sim \frac{1}{2\pi}\int_0^\infty \int_0^{2\pi} \frac{Q(r)}{G}\left[1-\exp\left(\frac{iGt}{2\rho_0 v_* \cos\varphi}\right)\right] r d\varphi dr \qquad (23)$$

where notation of the right-hand limit in the expression for G (17) can be omitted.

In the integrand (23) an item with exponential factor with a high parameter t can (for determination of w with the accuracy of items vanishing at $t \to \infty$) be replaced by its asymptotics in the vicinity of the point $r = r_*$, $\varphi = 0$. The immediate vicinity of the second singular point $r = r_*$, $\varphi = \pi$ determines the items that differ from the corresponding ones for $r = r_*$, $\varphi = 0$ only in the sign of the imaginary part (with the same real part). Thus, the result presents the doubled real part, determined by the immediate vicinity of the first mentioned singular point. In its immediate vicinity we obtain:

$$G \sim \alpha\psi^2 + \beta\varphi^2, \quad \cos\varphi \sim 1,$$
$$\psi = r - r_*, \quad \alpha = 6Dr_*^2, \quad \beta = \rho_0 r_* v_*^2 \qquad (24)$$

Replacing the accurate expressions by the above asymptotics we can neglect items vanishing at $t \to \infty$ and expand the limits of integration over up to infinity. We obtain for $t \to \infty$, $\psi \neq 0$:

$$\mathrm{Re}\int_0^{2\pi} \frac{1}{G}\exp\left(\frac{iGt}{2\rho_0 v_* \cos\varphi}\right) d\varphi$$
$$\sim 4\,\mathrm{Re}\, e^{i\alpha\psi^2 t}\int_0^\infty \frac{e^{i\beta\varphi^2 t/(2\rho_0 v_*)}}{\alpha\psi^2+\beta\varphi^2} d\varphi$$
$$= \frac{2\pi}{\sqrt{\alpha\beta|\psi|}}\left[1 - C\left(\frac{\tau\psi^2}{r_*^2}\right) - S\left(\frac{\tau\psi^2}{r_*^2}\right)\right], \qquad \tau = \frac{dr_*^2 t}{2\rho_0 v_*} \qquad (25)$$

Further:

$$\int_0^{2\pi} \frac{d\varphi}{G(r,\varphi)} = \frac{2\pi}{\sqrt{G_0^2(r) + \rho_0 r v^2 G_0(r)}}, \quad G_0(r) = G(r,0) \qquad (26)$$

Combining these expressions we can obtain displacement as:

$$w = \frac{1}{\sqrt{\alpha\beta}}\left[A + r_* \ell n\tau\right] + \varepsilon,$$
$$A = const, \quad \varepsilon \to 0, \quad (t \to \infty),$$
$$A = \int_0^{r_0}\left[\frac{\sqrt{\alpha\beta}\,\psi - \lambda_-}{\psi\lambda_-}(r_* - \psi) + \frac{\sqrt{\alpha\beta} - \lambda_+}{\psi\lambda_+}(r_* + \psi)\right] d\psi$$
$$+ 2r_*\int_0^\infty \frac{C(\psi^2) + S(\psi^2) - \frac{\psi}{1+\psi}}{\psi} d\psi, \qquad (27)$$

$$\lambda_\pm = \sqrt{\left[G_0(r_* \pm \psi)\right]^2 + \beta_0 G_0(r \pm \psi)}$$

(Note that the logarithm is increasing rather slowly, that is why in logarithmical asymptotics (27) constant A is not negligibly little as compared with $\ln\tau$.)

AN APPROXIMATE MODEL OF PLATE-LIQUID INTERACTION

It is rather difficult to calculate displacement $w(x, y)$ when $r \neq 0$ using relation (15) as neither of the two repeated integrals in the inversion formula can be found analytically. The left-hand part of (15) can be simplified by expansion of radical $æ$ in powers of K and q in the vicinity of the critical point $K = K_*$, $q = 0$ for $K > 0$ and $K = -K_*$, $q = 0$ for $K < 0$. Retaining only second powers we obtain:

$$\frac{K^2}{æ} = |K| - \frac{q^2}{2K_*} \qquad (28)$$

With such accuracy:

$$w(K,q) = \frac{Q(K,q)}{D\left(K^2+q^2\right)^2 + \rho_0 g - v^2\rho_0 A(K,q)},$$
$$A(K,q) = |K| - \frac{q^2}{2K_*} - i\varepsilon\,\mathrm{sgn}\,K,$$
$$\varepsilon = +0 \qquad (29)$$

The fundamental solution can be represented as a quadrature for any values of coordinates η, y (R, θ); for an integral over k or over φ($K = \rho\cos\varphi$, $q = \rho\sin\varphi$) can be evaluated analytically.

Further we use this simplification when solving a crack problem. So, to estimate an error caused by this approximation we consider the fundamental solution when $R = 0$ ($Q(K, q) = 1$). Turning to polar coordinates on the plane (K, q) we get ($R = 0$)

$$w(R,\theta) = \frac{1}{4\pi^2}\int_0^\infty\int_0^{2\pi} \frac{r d\varphi dr}{Dr^4 + \rho_0 g - v^2\rho_0 \bar{A}(r,\varphi)}$$
$$= \frac{1}{4\pi^2\sqrt{\rho_0 g D}}\int_0^\infty\int_0^{2\pi} \frac{\xi d\varphi d\xi}{\xi^4 + 1 - U^2\bar{B}(\xi,\varphi)},$$
$$\bar{A}(r,\varphi) = r|\cos\varphi| - \frac{r^2\sin^2\varphi}{2} - i\varepsilon\,\mathrm{sgn}(\cos\varphi),$$
$$\bar{B}(\xi,\varphi) = \xi|\cos\varphi| - \frac{\xi^2\sin^2\varphi}{2K_C} - i\varepsilon\,\mathrm{sgn}(\cos\varphi),$$
$$u = v/\sqrt{gL}, \quad K_c = K_* L, \quad \xi = rL, \quad L = (D/\rho_0 g)^{1/4} \qquad (30)$$

Then for $v < v_*$

$$w = \frac{1}{2\pi\sqrt{\rho_0 g D}}\int_0^\infty Q(\xi)\beta_1(\xi) d\xi \qquad (31)$$

and when $v > v_*$

$$w = \frac{1}{2\pi\sqrt{\rho_0 g D}}[A + B] \qquad (32)$$

where

$$\beta_1(\xi)=L_1[L_2+L_3],\quad \beta_2(\xi)=L_1[L_2+L_4],$$
$$L_1=\frac{2\xi}{\pi M\sqrt{\underline{B}(\xi)}},$$
$$M=\sqrt{2(\xi^4+1)K_C+u^2\xi^2+u^2\xi^2K_C},$$
$$L_{2,3}=\frac{\pm(u\xi-uK_C)+M}{\sqrt{\alpha(\xi)\pm\frac{u\xi}{K_C}M}}\operatorname{arctg}\frac{\sqrt{\underline{B}(\xi)}}{\sqrt{\alpha(\xi)\pm\frac{u\xi}{K_C}M}},$$
$$L_4=\frac{uK_C-u\xi+M}{2\sqrt{\frac{u\xi}{K_C}M-\alpha(\xi)}}\ln\left|\frac{\sqrt{\frac{u\xi}{K_C}M-\alpha(\xi)}-\sqrt{\underline{B}(\xi)}}{\sqrt{\frac{u\xi}{K_C}M-\alpha(\xi)}+\sqrt{\underline{B}(\xi)}}\right|,$$
$$A=\int_\Omega Q(\xi)\beta_1(\xi)\,d\xi,$$
$$B=\int_w Q(\xi)\beta_2(\xi)\,d\xi,$$
$$\alpha(\xi)=\frac{u^2\xi^2}{K_C}+\xi^4+1,\quad \underline{B}(\xi)=u^2\xi+\xi^4+1,$$
$$\xi\in\Omega,\ (\xi^4+1\ge u^2\xi);\quad \xi\in w,\ (\xi^4+1<u^2\xi)$$

As it is an accurate solution for subcritical velocity a derivative $\partial w/\partial x=0$ ($R=0$) and for overcritical one ($v>v_*$) when ($R=0$)

$$\frac{\partial w}{\partial x}=-\frac{1}{2\pi\sqrt{\rho_0 gD}}\int_w Q(\xi)\beta_3(\xi)\,d\xi \tag{33}$$

where

$$\beta_3(\xi)=\frac{2\left(z-\frac{K_C}{\xi}\right)K_C}{u^2z\sqrt{1-\left(z-\frac{K_C}{\xi}\right)^2}}$$
$$z=\sqrt{\frac{K_C^2}{\xi^2}+\frac{2K_C(\xi^4+1)}{u^2\xi^2}+1}$$

The velocity $\partial w/\partial t$ when $R=0$ is proportional to the power of radiation (a power developed by unit force), and in this steady-state problem it is simply connected with the derivative (33):

$$\frac{\partial w}{\partial t}=-v\frac{\partial w}{\partial x}$$

Comparison of exact and approximate results is given in Tables 1 and 2 $\left(u_*=v_*/\sqrt{gL}\simeq 1.325\right)$ of nondimensional values

$$w_0=2\pi\sqrt{\rho_0 gD}\,w,\quad w_1=-2\pi\sqrt{\rho_0 gD}\,\partial w/\partial x\quad \text{for } r=0$$

From the comparison it is obvious that with a 10% error, approximate description of plate-water interaction can be used in the range $0\le u\le 1.75=1.32u_*$, i.e., $0\le v\le 1.75\sqrt{gL}=1.32v_*$. For ice 1 m thick the upper bound corresponds to speed 21.4 m/s. For subcritical range, the error of displacement calculation is below 6%. Thus, for problems with a rather low range of velocities, if high accuracy is not required, it is possible to use the approximate model proposed above, which considerably simplifies the solution. It has been noted above that dynamical equations of plate-liquid interaction are of the convolution type; they can not be reduced to a differential equation (that means nonlocality of interaction). The approximation considered above corresponds to retaining the convolution character of the equation with respect to variable (along which a load is moving; more accurate, for which $\eta=x-vt$ is steady-state variable) and transition to differential dependence with respect to transversal variable y; as can be seen from Eq. 29, there exist zero, second ($-q^2$) and fourth (q^4) derivatives with respect to y in the equation. At the same time ratio $|K|$ corresponds to convolution with a generalized function $-\pi^{-1}\eta^{-2}$ (Slepian, 1990). Further we apply this approximation to a crack problem.

u	w_0 exact Eqs. (21),(22)	w_0 approx. Eqs. (31),(32)	error %
0.0	0.785	0.785	0.0
0.2	0.791	0.790	0.0
0.4	0.809	0.802	1.
0.6	0.843	0.827	2.
0.8	0.901	0.873	3.
1.0	1.006	0.963	4.
1.2	1.263	1.197	5.
1.4	1.168	1.075	8.
1.6	0.681	0.578	15.
1.8	0.488	0.387	21.
2.0	0.377	0.281	26.

Table 1 Approximate results of displacement

u	w_1 exact Eq. (22)	w_1 approx. Eq. (32)	error %
1.35	0.827	0.821	1.
1.4	0.797	0.781	2.
1.5	0.745	0.711	5.
1.6	0.701	0.652	7.
1.7	0.663	0.602	9.
1.8	0.630	0.560	11.
1.9	0.601	0.524	13.
2.0	0.576	0.493	14.

Table 2 Approximate results of $\partial w/\partial x$

CRACK PROBLEM: FORMULATION AND SOLUTION

Consider the ice cover with a straight semi-infinite crack, $\eta<0$, $y=0$. The following boundary conditions are given for the crack: breaking force:

$$D\left(\frac{\partial^3 w}{\partial y^3}+(2-v)\frac{\partial^3 w}{\partial\eta^2\partial y}\right)=P(\eta) \tag{34}$$

bending moment:

$$M=D\left(\frac{\partial^2 w}{\partial y^2}+v\frac{\partial^2 w}{\partial x^2}\right)=0,\quad (\eta>0) \tag{35}$$

Conditions of symmetry may be formulated at the prolongation of the crack line ($\eta>0$, $y=0$):

$$\frac{\partial w}{\partial y} = \frac{\partial^3 w}{\partial y^3} = 0 \tag{36}$$

We neglect a possible flow of liquid through the crack, i.e., assume that under boundary conditions for the potential Φ: $\partial\Phi/\partial z = \partial w/\partial t$ — no generalized function, located on the crack line, is added ($y = 0$, $\eta < 0$).

Due to the symmetry with respect to x axis, it follows that $\partial\Phi/\partial y = 0$, ($-\infty < \eta < \infty$, $y = 0$).

Further we make Laplace and cosine transforms of Eqs. 1 and 2 with respect to variables and y correspondingly (using the same notations).

With due regard to conditions (34-36) we can obtain (neglecting the inertia of the plate itself):

$$w(K,q) = \frac{Q(K,q) + D\left(w'''(K) - \left(q^2 + 2K^2\right)w'(K)\right)}{\Delta(K,q)} \tag{37}$$

and

$$\Delta(K,q) = D\left(K^2+q^2\right)^2 + \rho_0 g - \frac{\rho_0 v^2 K^2}{\sqrt{K^2+q^2}}$$

where $w'(K)$, $w'''(K)$ are Fourier transforms of $\partial w/\partial y$ and $\partial^3 w/\partial y^3$ with respect to η for $y = +0$. $Q(K, q)$ is the Fourier-Laplace transform with respect to η and y of external force $Q(\eta, y)$ (supplemental to the forces applied to the crack banks), which is assumed to be symmetrical with respect to η as $w''' = 0$ for $\eta > 0$. From the condition (34) it follows that:

$$w'''(K) = \frac{P(K)}{D} - (2-\nu)K^2 w'(K) \tag{38}$$

Denote:

$$w'(K) = F_-(K)L, \quad M(K)/D = F_+(K) \tag{39}$$

Here index "–" ("+") marks a function that is zero for $\eta > 0$ ($\eta < 0$).

From (37) it follows that:

$$w(K,y) = -\frac{2}{\pi}\int_0^\infty \frac{\left[Q(K,q) - P(K) + DF_-(K)L\left(q^2+\nu K^2\right)\right]}{\Delta(K,q)} \times \cos qy\, dq \tag{40}$$

Hence the moment for $\eta > 0$:

$$M = -\frac{2}{\pi}\int_0^\infty \frac{Q(K,q) + P(K) - DF_-L\left(q^2+\nu K^2\right)}{\Delta(K,q)} \times \left(q^2+\nu K^2\right)dq = F_+ \tag{41}$$

The integral in the right-hand part of Eq. 41 does not exist. That is connected with discontinuity of the derivative $\partial w/\partial y$ at $y = 0$; that is why the second derivative contains Dirac δ-function. To find the limit at $y = +0$, it is sufficient to subtract from the integrand F_-L. Thus, instead of (41) we obtain

$$\frac{2}{\pi}\int_0^\infty \left[\frac{Q(K,q) - P(K) - DLF_-\left(q^2+\nu K^2\right)}{\Delta(K,q)} \times \left(q^2+\nu K^2\right) - F_-L\right]dq = F_+$$

From this we come to the Wiener-Hopf equation with respect to transformations (passing to nondimensional variables by the formula 39 we consider $K' = KL$, $q' = qL$ and neglect the primes):

$$\begin{aligned}
&F_+ + S(K)F_- = \psi P^0(K) + \gamma, \\
&S(K) = \frac{2}{\pi}\int_0^\infty \left[\frac{\left(q^2+\nu K^2\right)^2}{\Delta^0(K,q)} - 1\right]dq, \\
&\psi(K) = \frac{2}{\pi}\int_0^\infty \frac{q^2+\nu K^2}{\Delta^0(K,q)}dq, \\
&\gamma = \frac{2}{\pi}\int_0^\infty \frac{Q^0(K,q)\left(q^2+\nu K^2\right)}{\Delta^0(K,q)}dq, \\
&\Delta^0(K,q) = \left(K^2+q^2\right)^2 + 1 + (\varepsilon + iKu)^2 \Big/ \sqrt{K^2+q^2}, \\
&P^0 = P\big/\left(\rho_0 g L^3\right), \quad Q^0 = Q\big/\left(\rho_0 g L^3\right), \quad \varepsilon = +0
\end{aligned} \tag{42}$$

To calculate S and ψ analytically, we use the above approximations, i.e., consider:

$$\Delta^0 \approx \Delta_\sim = \left(K^2+q^2\right)^2 + 1 - u^2\left(|K| - \frac{q^2}{2K_C}\right) - i\varepsilon \operatorname{sgn} K$$
$$\varepsilon = +0 \tag{43}$$

We determine the integrals by means of the residue theory for $v < v_*$:

$$S(K) = \frac{1}{\sqrt{d(K)}}\left[\varepsilon_R(K)R(K) - \varepsilon_I(K)I(K)\right] \tag{44}$$

$$\psi(K) = \frac{1}{\sqrt{d(K)}}\left[R(K) - \left(\alpha v^2 + (1-\nu)K^2\right)I(K)\right] \tag{45}$$

where

$$\begin{aligned}
&d(K) = K^4 + 1 - v^2|K|, \\
&\varepsilon_R(K) = \alpha v^2 + 2(1-\nu)K^2, \\
&\varepsilon_I(K) = m^2 + \alpha^2 v^4 + 2\alpha v^2(1-\nu)K^2 + (1-\nu)^2 K^4, \\
&R(K) = \frac{1}{\sqrt{2}}\sqrt{K^2 + \alpha v^2 + \sqrt{d(K)}}, \\
&I(K) = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{K^2 + \alpha v^2 + \sqrt{d(K)}}}, \\
&m(K) = \sqrt{\alpha^2 v^4 - 1 + v^2|K| + 2\alpha v^2 K^2}, \\
&\alpha = \frac{1}{4K_C}
\end{aligned}$$

So functions S and ψ have asymptotics at infinity $K \to \pm\infty$:

$$\begin{aligned}
&S(K) \sim \beta|K|, \\
&\psi(K) \sim \frac{1+\nu}{2}|K|^{-1}
\end{aligned} \tag{46}$$

where

$$\beta = (1-\nu)(3+\nu)/2$$

Using a routine technique for solving the Riemann problem (Muskhelishvili, 1966; Gakhov, 1980) and taking into account Eq. 46, we introduce the coefficient of Eq. 42 as a product. (Here the root branches are determined by relation $\sqrt{1}=1$, and singular points $K=\pm i0$ are related to the upper and lower half-spaces, respectively.)

$$S(K)=\beta\sqrt{a+iK}\sqrt{a-iK}\Lambda_-(K)\Lambda_+^{-1}(K),$$

$$a=\frac{S(0)}{\beta}$$

where factors β, $\sqrt{a-iK}$, $\sqrt{a+iK}$ are introduced so that $\Lambda_\pm$ have the following properties:

$$\begin{aligned}&\Lambda_-\ \Lambda_+^{-1}\to 1,\\ &\operatorname{Ind}\Lambda_-\ \Lambda_+^{-1}=0\end{aligned}\qquad (K\to\pm\infty)$$

So we obtain (Gakhov, 1980):

$$\Lambda_\pm(z)=\exp\left\{\frac{1}{2\pi i}\int_{-\infty}^{\infty}\ln\frac{S(\xi)}{\beta\sqrt{a^2+\xi^2}}\frac{d\xi}{z-\xi}\right\},$$

$$(\operatorname{Im} z \gtrless 0)$$

Then we can rewrite Eq. 42 in the next form:

$$\frac{F_+(K)\ \Lambda_+(K)}{\sqrt{a-iK}}+\beta F_-(K)\Lambda_-(K)\sqrt{a+iK}=$$

$$=P^0(K)\Lambda_+(K)\psi(K)/\sqrt{a-iK}$$

The solution of this problem can be introduced by formulae:

$$F_+(z)=\frac{\sqrt{a-iz}}{\Lambda_+(z)}U(z)$$

$$F_-(z)=-\frac{1}{\beta\Lambda_-(z)\sqrt{a+iz}}U(z) \tag{47}$$

where

$$U(z)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}P^0(K)\Lambda_+(K)\frac{\psi(K)}{\sqrt{a-iK}}\frac{dK}{K-z}$$

Formally, the solution of the crack problem can be closed; however, the following questions remain unanswered:

What level of the load does the steady-state solution correspond to?

What disposition of the crack and the load corresponds to the given level of the load?

To answer these questions we use the power criterion in the theory of cracks (Griffith's criterion). In the steady-state problem the flow of energy in the moving crack tip does not depend on the crack edge configuration, and it can be calculated as a work of bending moment (at the crack continuation) at the angle of crack side turning, it can be found from the solution in Eq. 47. (For steady-state problem, see Slepian, 1990.)

$$T=\lim_{z\to+\infty}\frac{D}{L}z^2\ F_+(iz)\ F_-(-iz)=\frac{D}{L}\hat{K}^2, \tag{48}$$

$$\hat{K}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{P^0(K)\psi(K)+\gamma(K)}{\sqrt{a-iK}}\Lambda_+(K)\,dK,$$

$$\Lambda_+(K)=\sqrt{\frac{\beta\sqrt{a^2+K^2}}{S(K)}}$$

$$\times\exp\left\{\frac{P.V.}{2\pi i}\int_{-\infty}^{\infty}\ln\frac{S(\xi)}{\beta\sqrt{a^2+\xi^2}}\frac{d\xi}{K-\xi}\right\}$$

So the answers to the above-stated questions can be found from power Griffith's criterion that can be described by an equality (Slepian, 1990):

$$T=2\gamma h$$

where γ is an effective surface energy (of the fracture) for the given velocity of movement.

Representing the load in the form:

$$P(\eta)=AP_0(\eta+\ell)$$

we see that the energy flow can be described as $T=A^2T_0(\ell)$. Propagation of the crack becomes possible only if:

$$A^2\max T_0(\ell)\geq 2\gamma h \tag{49}$$

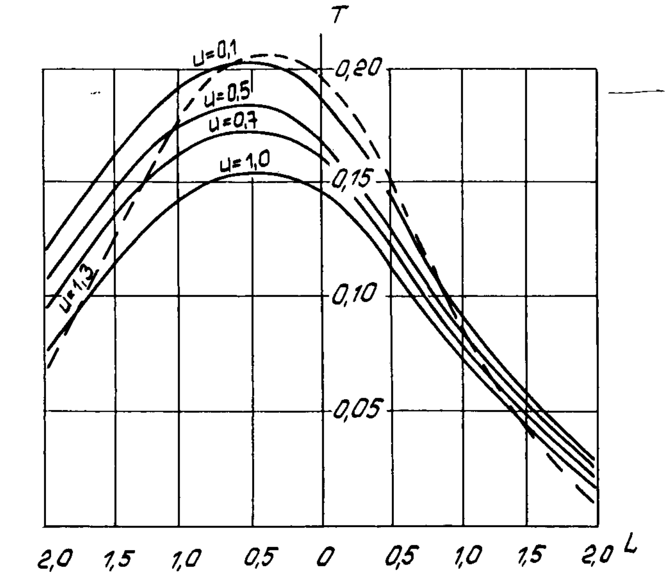

Fig. 1 Dependence of energy flux on the length between the force and the crack tip

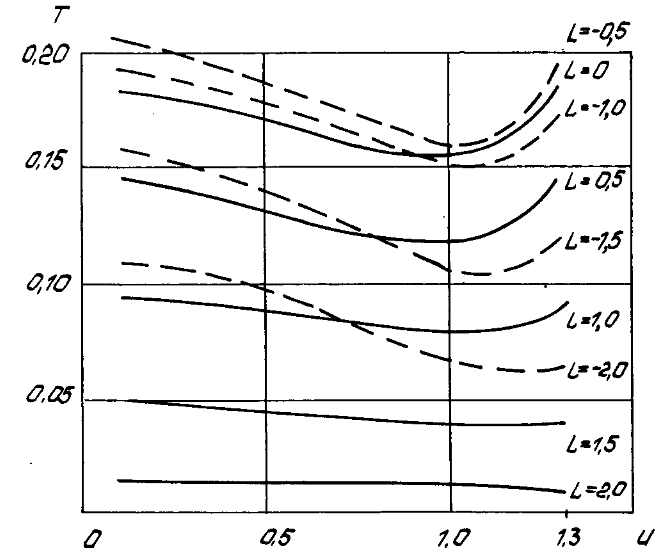

Fig. 2 Dependence of energy flux on velocity

Concerning the location of the load with respect to the crack (i.e., value of the parameter ℓ), note that, if inequality (49) is strict, ℓ is to be determined by the condition in Eq. 48. For low speed and for rather big amplitude A of the load, the parameter $\ell > 0$; negative value of ℓ is possible for big values of speed.

Eq. 48 is used for the calculation of energy flux for unit force concentrated in the moving point $\eta = L$, $y = 0$. Numerical results are represented at Figs. 1 and 2 for different values of speed u (for subcritical case $u < u_*$) and nondimensional distance L. It should be noted that maximum of energy flux is always reached in the case when the crack is moving in front of the force so that $L \approx -0.5$ (that makes approximately 9 m for the ice cover of 1 m thickness). It is worth emphasizing the next phenomenon: for constant L the energy flux at first decreases with an increase in speed and then increases when u approaches it critical value u_*.

The left branch of curves (Fig. 1) corresponds to stable crack propagation, and the right branch corresponds to unstable one.

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